Portfolio Credit Risk: Introduction

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Abstract

In the present technical report we examine the main theoretical aspects in some models used in Portfolio credit risk. We introduce the well-known Vasicek model, the large homogeneous portfolios or Vasicek distribution and their corresponding generalizations. An illustrative example considering factors following a logistic distribution is presented. Numerical experiments for several homogeneous portfolios are performed in order to compare these methods. Finally, we use the NAG Toolbox for $MATLAB^®$ for implementing prototypes of these models quickly.

1   Introduction

Credit risk is a critical area in banking and is of concern to a variety of stakeholders: financial institutions, consumers and regulators. Credit risk is the risk of loss resulting from an obligor's inability to meet its legal obligation according to the debt contract. This circumstance is called the default event.

For financial institutions it is essential to quantify the credit risk at a portfolio level. Portfolio credit loss modelling requires the default dependence among obligors. A common approach is utilizing one or multiple factor models, such as the obligors are independent conditional on some latent common factors, which in some cases are assumed to follow a standard Normal distribution. Some common factors are the state of the economy, changes of a market index or interest rates, for instance. Furthermore, it is usually considered that an obligor might incur in default in a fixed time. Financial institutions are also interested in the computation of common risk measures, such as Value-at-Risk (VaR) and Expected Shortfall (ES), which are typically used to determine the reserve capital to cover potential extreme losses. VaR is the measure chosen in the Basel II Accord (Basel Committee on Bank Supervision).

Credit rating agencies such as Moody's, Standard & Poor's and Fitch focus their business in credit risk management and are particularly interested in the portfolio loss distribution. Credit ratings represent the creditworthiness of individual corporations and consumers. The rating notes are classified by letters, Aaa and Baa are examples of Moody's rating system, which uses Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C to represent the likelihood of default from the lowest to the highest. The speculative-grade corporate bonds are sometimes said to be high-yield or junk.

2   Vasicek Model

The general approach in the factor model is that the default times in a portfolio are driven by several risk factors. The portfolio is composed by N debtors whose default is driven by

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the variation of its firm's value. The value of the assets of the nth debtor at time t is denoted by $V_n(t)$. Debtor n defaults if its firm's value falls below a default threshold $V_n(T) \leq \alpha_n$. The asset values of different debtors are assumed to be correlated with each other and the variance-covariance matrix of the V_i, $i \in N$ is denoted by Σ.

The covariance matrix Σ has N^2 elements but is symmetric and may be defined by the $\frac{1}{2}N(N-1)$ elements of the upper triangular part, which characterize the joint default probability between the N debtors. In order to simplify and reduce computational effort, M factors are chosen reducing the matrix dimensions to $N \times M$. If a single factor is considered, the number of coefficients is equal to N (one-factor model), which is reduced to just one parameter under the assumption of equal correlation coefficient. To clarify the previous concepts, a more detailed situation that will be used as a framework for the next theoretical aspects is presented.

Definition 2.1 *Consider a portfolio of N assets (structured finance products, for instance) where the nth asset has an initial value of V_n and its default probability p_n over a known time horizon T_i for which the portfolio's default distribution is determined. Some assumptions are considered in order to simplify the model:*

1. *Each asset in the portfolio corresponds to a different debtor. This is modelled if assets belonging to the same debtor are aggregated.*

2. *All assets have bullet amortisation (i.e. The payment of the principal balance is due at the end of the loan term).*

Note that in this framework we can still calibrate the model to reflect different individual default probabilities p_n over the time horizon by setting the barrier level to that level which replicates the given individual default probability. Assuming $V_n(t)$ is normally distributed, this level is

$$\alpha_n = \Phi^{-1}(p_n) \tag{1}$$

where $\Phi(\cdot)$ is the cumulative Normal distribution function.

2.1 The distributions of defaults

The values of the assets of the obligors are driven by a common *systematic* factor Y and an *idiosyncratic* factor ϵ_n:

$$V_n(t) = \sqrt{\rho_n}\, Y + \sqrt{1-\rho_n}\, \epsilon_n \tag{2}$$

where Y and ϵ_n, $n \leq N$ are independent normally distributed random variables with mean 0 and variance 1 and $\rho_n \in [0,1]$. Using this approach the values of the assets of two obligors n and $m \neq n$ are correlated with linear correlation coefficient ρ. The important point is that *conditional on the realisation of the systematic factor Y, the firm's values and the defaults are independent.*

The systematic factor Y can be viewed as an indicator of the state of the business cycle, and the idiosyncratic factor ϵ_n as a firm-specific effects factor. The threshold α_n of the firm is mainly determined by the firm's reserves and balance-sheet structure. The relative sizes of the idiosyncratic and systematic components are controlled by the correlation coefficient ρ. If $\rho = 0$, then the business cycle has no influence on the fates of the firms, if $\rho = 1$, then it is the only driver of defaults, and the individual firm has no control whatsoever.

For example: first, the business cycle variable Y materialises, and conditional on the general state of the economy, the individual defaults occur independently from each other, but with a default probability $p_n(y)$ which depends on the state of economy. This default probability is

$$p_n(y) = \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1-\rho_n}}\right) \tag{3}$$

Indeed, the individual conditional default probability $p(y)$ is the probability that the firm's value $V_n(t)$ is below the threshold α_n, given that the systematic factor Y takes the value y:

$$
\begin{aligned}
p_n(y) &= P[V_n(t) < \alpha_n \,|\, Y = y] \tag{4}\\
&= P[\sqrt{\rho_n}\, y + \sqrt{1 - \rho_n}\, \epsilon_n < \alpha_n]\\
&= P\left[\epsilon_n < \frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right]\\
&= \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right)
\end{aligned}
$$

It can be proved that the expected value of the random variable $p_n(y)$ is the default probability p_n. See proof in [2, subsection 4.4.1]. This result remarks the fact that the expected value of the default probability conditional to a realisation of Y does not depend on the factor distribution; therefore, this statement holds for any factor which is not normally distributed. In the case that the assumption about the factor distribution was non Normal (NIG distribution, for instance), one must carefully determine the default threshold α_n since $\alpha_n = \Phi^{-1}(p_n)$ does not apply. The default threshold is obtained by solving the following equation for α_n, where $\phi(\cdot)$ denotes the density function for the factor:

$$
E[p_n(y)] = \int_{-\infty}^{\infty} \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right) \phi(y)\, dy = p_n \tag{5}
$$

The next step is to obtain the distribution function of the defaults. The probability of having exactly n defaults is the average of the conditional probabilities of n defaults, averaged over the possible realisations of Y and weighted with the probability density function $\phi(y)$,

$$
P[X = n] = \int_{-\infty}^{\infty} P[X = n \,|\, Y = y] \phi(y)\, dy \tag{6}
$$

Conditional on the realisation $Y = y$ of the systematic factor, the probability of having n defaults is given by the binomial probability mass function, which is a discrete probability function, since defaults are discrete random variables,

$$
P[X = n \,|\, Y = y] = \binom{N}{n} (p(y))^n (1 - p(y))^{N-n} \tag{7}
$$

where we use the conditional independence of the defaults in the portfolio. Substituting this and (4) into Equation (6) we obtain that the probability of having exactly n defaults in the underlying portfolio is given by

$$
P[X = n] = \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right)\right)^n \left(1 - \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right)\right)^{N-n} \phi(y)\, dy \tag{8}
$$

Therefore, the default distribution function, under the normality assumption is,

$$
P[X \le m] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} \left(\Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right)\right)^n \left(1 - \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right)\right)^{N-n} \phi(y)\, dy \tag{9}
$$

2.2 Computing the default probability function

The method used to evaluate the previous integrals is the Gauss–Hermite Quadrature. The Gauss–Hermite quadrature is a form of Gaussian quadrature for approximating the value of integrals of the next form,

$$
\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \approx \sum_{i=1}^{k} w_i f(x_i) \tag{10}
$$

Where k is the number of nodes used. The abscissae are given by the roots x_i of the Hermite polynomials $H_k(x)$ and the associated weights w_i are given by

$$w_i^{GH} = \frac{2^{k-1} k! \sqrt{\pi}}{k^2 [H_{k-1}(x_i)]^2}, \quad i = 1, 2, \ldots, k \tag{11}$$

In order to compute the distribution function of the defaults with this method, a reformulation is required:

$$P[X \leq m] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} h(y) \, dy$$

Let us take $h(y)$ as

$$h(y) = \left(\Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}} \right) \right)^n \left(1 - \Phi\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}} \right) \right)^{N-n}$$

We need the following change of variable,

$$x^2 = \frac{y^2}{2} \Leftrightarrow y = x\sqrt{2}$$

apply integration by substitution

$$P[X \leq m] = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2} h(x\sqrt{2}) \sqrt{2} dx = \sum_{n=0}^{m} \binom{N}{n} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} h(x\sqrt{2}) dx$$

and finally with (10) and (11)

$$P[X \leq m] \approx \sum_{n=0}^{m} \binom{N}{n} \frac{1}{\sqrt{\pi}} \sum_{i=1}^{k} w_i^{GH} \left(\Phi\left(\frac{\alpha_n - \sqrt{2\rho_n}\, x_i}{\sqrt{1 - \rho_n}} \right) \right)^n \left(1 - \Phi\left(\frac{\alpha_n - \sqrt{2\rho_n}\, x_i}{\sqrt{1 - \rho_n}} \right) \right)^{N-n} \tag{12}$$

Similarly, we can transform the probability distribution of the defaults, leading to the following expression

$$P[X = n] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{k} w_i^{GH} \binom{N}{n} \left(\Phi\left(\frac{\alpha_n - \sqrt{2\rho_n}\, x_i}{\sqrt{1 - \rho_n}} \right) \right)^n \left(1 - \Phi\left(\frac{\alpha_n - \sqrt{2\rho_n}\, x_i}{\sqrt{1 - \rho_n}} \right) \right)^{N-n} \tag{13}$$

In order to easily check the obtained results we consider six homogeneous portfolios with 100 assets and with $p_n = p = 0.15$, common correlation between assets $\rho_n = \rho$ and all loans having the same size. We use these three measures for comparison:

1. Expected Loss (EL): $EL(L) = E[L]$ where L is the Portfolio Loss at time T.

$$E[L] = \langle l^s, p^s \rangle \tag{14}$$

   where l^s is the vector of losses for each scenario and p^s is the corresponding vector of probabilities. In the homogeneous portfolio the mean of the distribution is $EL = p$, therefore, this is a key measure for checking the quality of the method.

2. Portfolio standard deviation: $Stdev[L] = \sqrt{E[L^2] - E[L]^2}$

3. VaR_β: Value-at-Risk with confidence level $\beta = 99.9\%$. The β-quantile of the loss distribution $F(x)$

$$VaR_\beta(X) = inf\{x, F(x) \geq \beta\} = F^{-1}(\beta) \tag{15}$$

   VaR is a risk measure widely used to quantify the risk. This is the measure chosen in the Basel II Accord for the computation of capital requirements, meaning that a bank that manages its risks according to Basel II must reserve capital by an amount of $x_\beta = F^{-1}(\beta)$ to cover potential extreme losses.

In Table 1 we can notice that the number of correct figures diminishes as we consider more extreme cases, $\rho = 0.3$ or $\rho = 0.5$. We have included a metric called *standard deviation-over-mean ratio*, which is used to calibrate the factor loadings ($w = \sqrt{\rho}$) for ABS (Asset-Backed-Securities) and MBS (Mortgage-Backed-Securities).
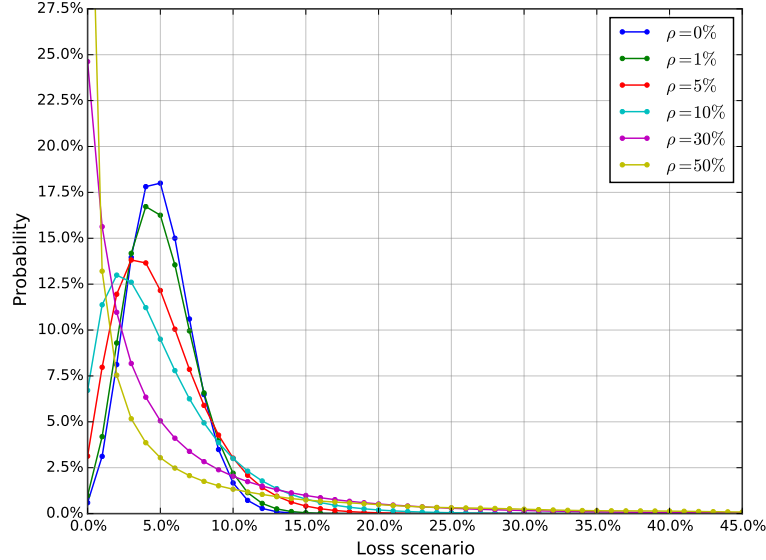

Figure 1: Default probabilities distributions under different asset correlation. Parameters: $N = 100$ and $p = 5\%$. Asset correlation ρ in percentage points.

ρ	Mean	Stdev	Stdev/Mean	$VaR_{0.999}$
0	0.050000	0.021794	0.4359	0.1346
0.01	0.050000	0.024119	0.4824	0.1478
0.05	0.050000	0.032215	0.6443	0.2028
0.10	0.050000	0.040935	0.8187	0.2716
0.30	0.049999	0.071157	1.4231	0.5570
0.50	0.049997	0.100325	2.0066	0.8145

Table 1: Default probabilities distributions computed using a Gauss–Hermite quadrature with 20 nodes.

3 Generalized Vasicek Model

In this section we focus on the computation of the loss distribution with a general distribution function. Schönbucher in [3] proposed the generalized one-factor model.

Assumption 3.1 (Generalized One-Factor Model) *The values of the assets of the obligors are driven by a common factor Y which has distribution function $G(y)$, and a idiosyncratic noise component ϵ_n which is distributed according to the distribution function $H(\epsilon_n)$*

$$V_n(t) = \sqrt{\rho_n}\, Y + \sqrt{1 - \rho_n}\, \epsilon_n \quad n \leq N \tag{16}$$

where $Y \sim G$ and ϵ_n, $n \leq N$ are independent and identically $H(\epsilon_n)$-distributed with mean 0 and variance 1 and $\rho \in [0, 1]$.

Therefore, the generalized default probability or conditional default risk is given by

$$p_n(y) = H\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right) \tag{17}$$

and must satisfy

$$E[p_n(y)] = \int_{-\infty}^{\infty} H\left(\frac{\alpha_n - \sqrt{\rho_n}\, y}{\sqrt{1 - \rho_n}}\right) G'(y)\, dy = p_n \tag{18}$$

For the sake of simplicity, and without loss of generality, let us assume $p_n = p$ for all $n \in \{1, \ldots, N\}$, and consequently, $p_n(y) = p(y)$ for all $n \in \{1, \ldots, N\}$. Furthermore, let us consider a portfolio with a common asset correlation ρ, such that $\rho_n = \rho$ for all $n \in \{1, \ldots, N\}$. We first need a numerical method for solving the previous equation for $\alpha_n = \alpha$ and finally we compute the generalized default probability function given by

$$P[X = n] = \int_{-\infty}^{\infty} \binom{N}{n} \left(H\left(\frac{\alpha - \sqrt{\rho}\, y}{\sqrt{1 - \rho}}\right)\right)^n \left(1 - H\left(\frac{\alpha - \sqrt{\rho}\, y}{\sqrt{1 - \rho}}\right)\right)^{N-n} G'(y)\, dy \tag{19}$$

For some cases (18) can be easily solved for α, for example when considering the Vasicek one-factor Gaussian model. In this special case we can make use of the following identity,

$$\int_{-\infty}^{\infty} \Phi\left(\frac{z - \mu}{\sigma}\right) \phi(z)\, dz = \Phi\left(\frac{-\mu}{\sqrt{\sigma^2 + 1}}\right) \tag{20}$$

$$E[p_n(y)] = \Phi\left(-\frac{-\alpha/\sqrt{\rho}}{(\sqrt{1 - \rho/\sqrt{\rho}})^2 + 1)^{1/2}}\right) = \Phi\left(\frac{\alpha}{\sqrt{1 - \rho/\rho + 1}\sqrt{\rho}}\right) = \Phi\left(\frac{\alpha}{\sqrt{1/\rho}\sqrt{\rho}}\right)$$
$$= \Phi(\alpha) = p_n \Leftrightarrow \alpha = \Phi^{-1}(p)$$

Hence, in the Vasicek one-factor Gaussian model, α only depends on the individual probability of default p, for example $\alpha = \Phi^{-1}(0.15) = -1.036433389$. For other statistical distributions more sophisticated methods are required, since there might not exist analytical expressions. Instead, a numerical method based on the discretization of the integral in (18) by using the double-exponential transformation can be applied (see [1, 4]). Then α is obtained by using a root-finding algorithm (e.g. Bisection, Newton-Raphson or Brent (`nag_roots_contfn_brent`, `c05ay`)). Further details can be found in [2, subsection 4.2.1].

3.1 Special factor models

Until now, we have computed portfolios assuming a systematic factor and idiosyncratic factor both normally distributed. In order to extend the one-factor model, we test the generalized Vasicek model for other continuous distributions with support $x \in \mathbb{R}$. Thus, we choose the logistic distribution as an illustrative example and we refer to this model as the *Logistic-factor model*.

The probability density function and the cumulative distribution function of the logistic distribution are given by,

$$f(y; \mu, s) = \frac{e^{-\frac{y-\mu}{s}}}{s(1 + e^{-\frac{y-\mu}{s}})^2} = \frac{1}{4s}\operatorname{sech}^2\left(\frac{y-\mu}{2s}\right) = \frac{1}{4s\cosh^2\left(\frac{y-\mu}{2s}\right)} \tag{21}$$

$$F(y; \mu, s) = \frac{1}{1 + e^{-\frac{y-\mu}{s}}} = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{y-\mu}{2s}\right) \tag{22}$$

As previously stated, Y and ϵ_n are centred and standardized random variables, then we have to choose the parameters for each factor such that they have zero mean and unit variance.

$$E[Y] = \mu = 0; \quad Var[Y] = 1 \Leftrightarrow \frac{s^2 \pi^2}{3} = 1 \Leftrightarrow s = \frac{\sqrt{3}}{\pi}. \tag{23}$$

Substituting (21) and (22) in Equation (18), we obtain the following equation to be solved for α,

$$\frac{\pi}{8\sqrt{3}} \int_{-\infty}^{\infty} \left(1 + \tanh\left(\frac{\pi(\alpha - \sqrt{\rho}\, y)}{2\sqrt{3(1-\rho)}}\right)\right) \text{sech}^2\left(\frac{y\pi}{2\sqrt{3}}\right) dy = p \tag{24}$$

We consider homogeneous portfolios in order to compare the result to large homogeneous portfolio models. Given $p = 0.15$ and $\rho = 0.1$, $\alpha = -0.970082643277$. Once the threshold α is computed, we just need to evaluate the following integral for each asset $n \in \{1, \ldots, N\}$, which after some steps has the following form,

$$P[X = n] = \frac{\pi}{2^{N+2}\sqrt{3}} \int_{-\infty}^{\infty} \binom{N}{n} \left(1 + \tanh\left(\frac{\pi(\alpha - \sqrt{\rho}\, y)}{2\sqrt{3(1-\rho)}}\right)\right)^n \left(1 - \tanh\left(\frac{\pi(\alpha - \sqrt{\rho}\, y)}{2\sqrt{3(1-\rho)}}\right)\right)^{N-n}$$

$$\times \text{sech}^2\left(\frac{y\pi}{2\sqrt{3}}\right) dy \tag{25}$$

This integral cannot be directly evaluated using Gauss–Hermite quadrature, therefore we need a general purpose integration routine. We could use several methods, such as tanh-sinh quadrature (double exponential quadrature) or Monte Carlo using Median Latin hypercube sampling are described in [2]. As shown in Table 2, portfolio loss distributions highly skewed and leptokurtic (fat tails) are more difficult to compute, therefore some loss of digits is expected.
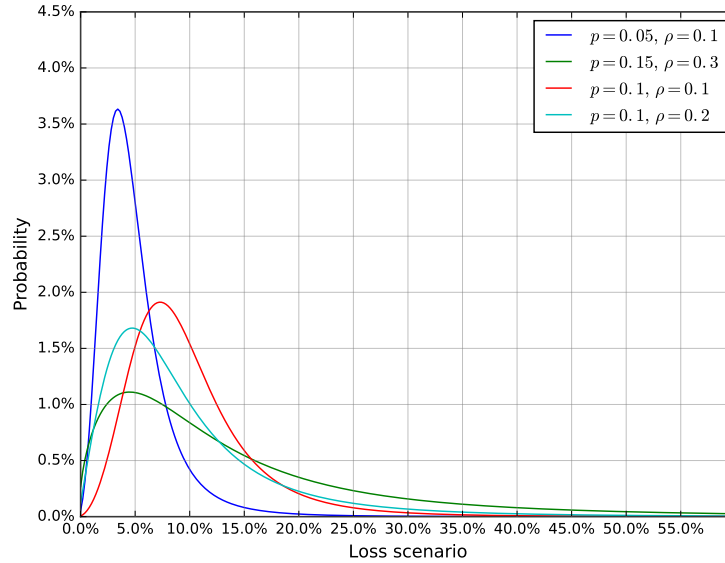

Figure 2: Generalized Vasicek one-factor model. Default probabilities distributions under different asset probability of default and correlation. Parameters: $N = 500$, $p \in \{0.05, 0.1, 0.15\}$ and $\rho \in \{0.1, 0.2, 0.3\}$.

Parameters	Mean	Stdev	$VaR_{0.999}$
$p = 0.05$, $\rho = 0.10$	0.050000	0.033978	0.2855
$p = 0.15$, $\rho = 0.30$	0.149996	0.143692	0.8860
$p = 0.10$, $\rho = 0.10$	0.100000	0.059033	0.4582
$p = 0.10$, $\rho = 0.20$	0.100000	0.088145	0.6638

Table 2: Main portfolio metrics for generalized portfolios (Logistic-factor model).

As the number of assets in the portfolio increases, the number of quadratures needed to obtain the same level of accuracy increases accordingly; being computationally expensive. For larger portfolios the double exponential quadrature (applicable when the integrand is analytic) performs better, since the computation of nodes is faster and is able to handle infinite integration intervals better. We have included a very simplified implementation in section 7.

4   The Large Portfolio Approximation

In this section we introduce the so-called Vasicek Single Factor Model for a large portfolio of homogeneous assets. This model was initially introduced by Vasicek in [5] and generalized by Schönbucher in [3]. The Single Factor Model increases the tractability of the one-factor model by assuming a portfolio of homogeneous assets ($p_n = p$ and $\rho_n = \rho$) composed by a number of obligors tending to infinity, given an explicit formula for the default distribution which is directly derived from the Law of Large Numbers.

4.1   Properties of the Vasicek distribution

Several assumptions are considered in the Single Factor Model:

Assumption 4.1 (The number of obligors $N \to \infty$) *Due to individual defaults being independent when conditioned to the realization of the common factor Y, by the Law of Large Numbers, the average of N fractions of defaulted obligors in the portfolio sharing the same probability X_n converges towards the individual default probability of each individual obligor $p(Y)$.*

For a given realization of the systematic factor $Y = y$, the individual default probability is given by

$$E\left[\sum_{n=1}^{N} \frac{X_n}{N} \,\Big|\, Y = y\right] = p(y) = \Phi\left(\frac{\alpha - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \tag{26}$$

Assumption 4.2 (The loss given default is deterministic and homogeneous) *The loss of each obligor due to default is expressed as a percentage of its size and is common to all obligors in the portfolio.*

Assumption 4.3 (The contributions of each obligor are similar) *All the obligors in the portfolio have the same relative size $s_n = 1/N$, the same correlation $\rho_n = \rho$ and the same default threshold $\alpha_n = \alpha$. The convergence of the portfolio loss distribution actually holds for unequal relative sizes if the portfolio contains a sufficiently large number of obligors without it being dominated by a few obligors much larger than the rest, consequently*

$$\sum_{n=1}^{N} s_n^2 \to 0 \tag{27}$$

As a consequence of these three assumptions the cumulative distribution function of portfolio

losses can be constructed as follows

$$P[p(y) \leq x] = P\left(\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}\,y}{\sqrt{1-\rho}}\right) \leq x\right)$$

$$= P\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}\,y}{\sqrt{1-\rho}} \leq \Phi^{-1}(x)\right)$$

$$= P\left(-y \leq \frac{\Phi^{-1}(x)\sqrt{1-\rho} - \Phi^{-1}(p)}{\sqrt{\rho}}\right)$$

$$= \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right)$$

The portfolio loss distribution given by the cumulative distribution function is a continuous distribution concentrated on the interval $x \in [0,1]$

$$F(x;p,\rho) = P[p(y) \leq x] = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) \tag{28}$$

which possesses a symmetry property

$$F(x;p,\rho) = 1 - F(1-x;1-p,1-\rho) \tag{29}$$

The portfolio loss density function or Vasicek distribution is defined as a two-parametric $(0 < p < 1$ and $0 < \rho < 1)$ continuous distribution given by the derivative of the portfolio loss distribution

$$f(x;p,\rho) = \frac{\partial F(x;p,\rho)}{\partial x}$$

$$= \sqrt{\frac{1-\rho}{\rho}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sqrt{1-\rho}\,\Phi^{-1}(x)-\Phi^{-1}(p))^2}{2\rho}}\frac{\partial}{\partial x}\Phi^{-1}(x)$$

$$= \sqrt{\frac{1-\rho}{\rho}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sqrt{1-\rho}\,\Phi^{-1}(x)-\Phi^{-1}(p))^2}{2\rho}}\sqrt{2\pi}e^{\frac{(\Phi^{-1}(x))^2}{2}}$$

$$= \sqrt{\frac{1-\rho}{\rho}}e^{\left(-\frac{(\sqrt{1-\rho}\,\Phi^{-1}(x)-\Phi^{-1}(p))^2}{2\rho}+\frac{(\Phi^{-1}(x))^2}{2}\right)} \tag{30}$$

4.1.1   Moments of the Vasicek distribution

Proposition 4.4 *Let n be a positive integer and $\xi_1, \ldots, \xi_n$ i.i.d. standard normal. If X is Vasicek-distributed with parameters $p = \Phi(\alpha)$ and ρ, then*

$$E[X^n] = E\left[\Phi\left(\frac{\alpha - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right] = E\left[\prod_{i=1}^{n}P[\sqrt{\rho}Y + \sqrt{1-\rho}\xi_i \leq \alpha|Y]\right]$$

$$= P[Y_1 \leq \alpha, \ldots, Y_n \leq \alpha]$$

where $(Y_1, \ldots, Y_n)$ is a multi-variate normal vector with $E[Y_i] = 0$, $Var[Y_i] = 1$ and $corr[Y_i, Y_j] = \rho$, $i \neq j$.

Thus mean and variance expressions are:

$$E[X] = P[Y \leq \alpha] = p \tag{31}$$

$$var[X] = E[X^2] - E[X]^2 = P[Y_1 \leq \alpha, Y_2 \leq \alpha] - p^2 = \Phi_2(\alpha, \alpha, \rho) - p^2 \tag{32}$$

where (Y_1, Y_2) is a bivariate normal vector with $E[Y_i] = 0$, $Var[Y_i] = 1$ and $corr[Y_1, Y_2] = \rho$, $i = 1, 2$ and $\Phi_2(\cdot, \cdot, \rho)$ is the bivariate cumulative normal distribution function with zero mean and variance equal to the correlation ρ.

Finally, the quantile function or inverse cumulative distribution function can be easily obtained from Equation (28) as follows

$$\Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) = \beta \implies \sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p) = \sqrt{\rho}\Phi^{-1}(\beta)$$

$$F^{-1}(\beta; p, \rho) = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\beta)}{\sqrt{1-\rho}}\right) \tag{33}$$

which can be used for the calculation of Value-at-Risk.

4.2   Numerical example

Table 3 shows the relative error due to the use of the approximation. Observe, that the error is reduced for skewed and leptokurtic portfolios. In many applications the order of magnitude of the errors is acceptable, therefore the large homogeneous approximation is significantly useful if the characteristics of the assets within a portfolio are similar.
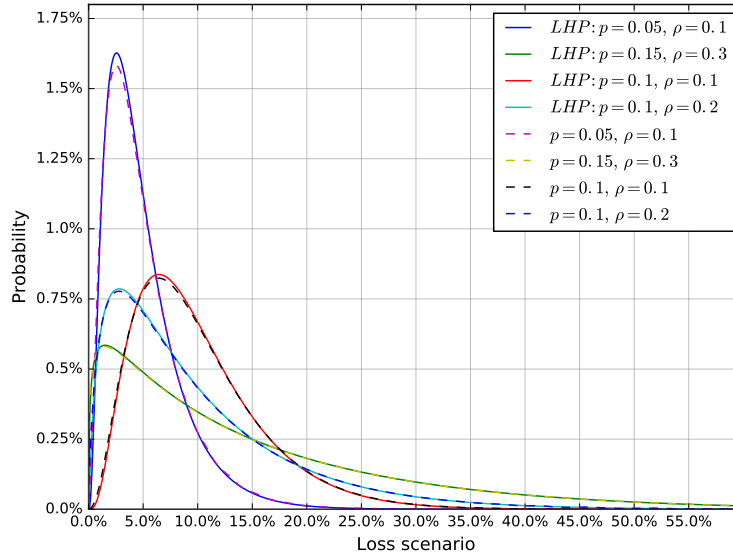

Figure 3: Large homogeneous portfolio vs Vasicek one-factor model. Default probabilities distributions under different asset probability of default and correlation. Parameters: $N = 1000$, $p \in \{0.05, 0.1, 0.15\}$ and $\rho \in \{0.1, 0.2, 0.3\}$.

Parameters	Mean	Stdev	$VaR_{0.999}$
$p = 0.05,\ \rho = 0.10$	0.050000 (**0.0%**)	0.035483 (**1.9%**)	0.2441 (**1.3%**)
$p = 0.15,\ \rho = 0.30$	0.150000 (**0.0%**)	0.138304 (**0.3%**)	0.7861 (**0.3%**)
$p = 0.10,\ \rho = 0.10$	0.100000 (**0.0%**)	0.058499 (**1.3%**)	0.3777 (**0.9%**)
$p = 0.10,\ \rho = 0.20$	0.100000 (**0.0%**)	0.085317 (**0.6%**)	0.5475 (**0.5%**)

Table 3: Relative error (within parenthesis) with respect to Vasicek one-factor model.

In terms of computational time[1], the large homogeneous portfolio is roughly 150 times faster than the Vasicek one-factor model in our implementation of the NAG Toolbox for $MATLAB^{\circledR}$.

---

[1]Intel® Core™ i5-5300U CPU @ 2.30GHz using MATLAB® R2015b.

5 General Loss Distribution

The Vasicek distribution implies a strong assumption; the systematic and idiosyncratic factors are normally and standardized distributed. This fact is far from common and in general we encounter factors with different distributions. Schönbucher provided a general portfolio cumulative distribution function in [3] given by

$$F(x; \alpha, \rho) = P[X \le x] = 1 - G\left(\frac{\alpha}{\sqrt{\rho}} - \sqrt{\frac{1-\rho}{\rho}} H^{-1}(x)\right) \tag{34}$$

We can derive Equation (34) to obtain the general portfolio loss density function

$$f(x; \alpha, \rho) = \frac{\partial F(x; \alpha, \rho)}{\partial x} = \sqrt{\frac{1-\rho}{\rho}} G'\left(\frac{\alpha}{\sqrt{\rho}} - \sqrt{\frac{1-\rho}{\rho}} H^{-1}(x)\right) \frac{\partial}{\partial x} H^{-1}(x) \tag{35}$$

and the quantile function or inverse cumulative distribution function can be easily obtained from Equation (34)

$$F^{-1}(\beta; \alpha, \rho) = H\left(\frac{\alpha - \sqrt{\rho} G^{-1}(1-\beta)}{\sqrt{1-\rho}}\right) \tag{36}$$

Now we proceed to generate explicit formulas for the large homogeneous approximation of the *Logistic-factor model*.

5.1 Logistic-factor model approximation

The *Logistic-factor model* $(L-L)$ is defined as $Y \sim Logistic(0, \frac{\sqrt{3}}{\pi})$ and $H \sim Logistic(0, \frac{\sqrt{3}}{\pi})$, as previously stated. For this model we need to compute the derivative of the quantile function of the logistic distribution

$$F(x; \mu, s) = \frac{1}{1 + e^{-\frac{x-\mu}{s}}} \implies F^{-1}(p; \mu, s) = \ln\left(\frac{p}{1-p}\right) s + \mu, \tag{37}$$

$$\frac{\partial}{\partial p} F^{-1}(p; s) = \frac{s}{p(1-p)} \tag{38}$$

The $L-L$ distribution function is obtained after a few steps

$$F(x; \alpha, \rho) = 1 - \frac{1}{2}\left(1 + \tanh\left(\frac{\pi}{2\sqrt{3}\rho}(\alpha - \sqrt{1-\rho}\ln\left(\frac{x}{1-x}\right)\frac{\sqrt{3}}{\pi})\right)\right)$$
$$= \frac{1}{2}\left(1 - \tanh\left(\frac{\alpha\pi}{2\sqrt{3}\rho} - \frac{\sqrt{1-\rho}\ln\left(\frac{x}{1-x}\right)}{2\sqrt{\rho}}\right)\right) \tag{39}$$

and its derivative leads to the $L-L$ density function

$$f(x; \alpha) = \sqrt{\frac{1-\rho}{\rho}}\frac{\pi}{4\sqrt{3}}\text{sech}^2\left(\frac{\pi}{2\sqrt{3}}\left(\frac{\alpha}{\sqrt{\rho}} - \sqrt{\frac{1-\rho}{\rho}}\ln\left(\frac{x}{1-x}\right)\frac{\sqrt{3}}{\pi}\right)\right)\frac{\sqrt{3}}{\pi(1-x)x}$$
$$= \sqrt{\frac{(1-\rho)}{16\rho}}\text{sech}^2\left(\frac{\alpha\pi}{2\sqrt{3}\rho} - \frac{\sqrt{1-\rho}\ln\left(\frac{x}{1-x}\right)}{2\sqrt{\rho}}\right)\frac{1}{(1-x)x} \tag{40}$$

Finally, we apply the derivative of the quantile function of the logistic distribution to obtain the $L-L$ quantile function

$$F^{-1}(\beta; \alpha, \rho) = \frac{1}{2}\left(1 + \tanh\left(\frac{\pi}{2\sqrt{3}}\frac{\alpha - \sqrt{\rho}\ln\left(\frac{1-\beta}{\beta}\right)\frac{\sqrt{3}}{\pi}}{\sqrt{1-\rho}}\right)\right) \tag{41}$$

As shown in Table 4, as the number of assets increases, the approximation by using the general loss distribution improves. As previously stated, without very strict constraints in terms of accuracy, it is worth including this approximation in your toolbox for credit risk, since for the generalized portfolio the speedup is very significant. Furthermore, this method can be applied to other statistical distribution with several parameters. Examples for the exponentially-modified Gaussian distribution and the normal inverse Gaussian distribution can be found in [2]. The usage of statistical distributions with more than two parameters implies that we obtain extra free parameters, which increase the flexibility of the distribution to fit the factor data (e.g. stock returns).
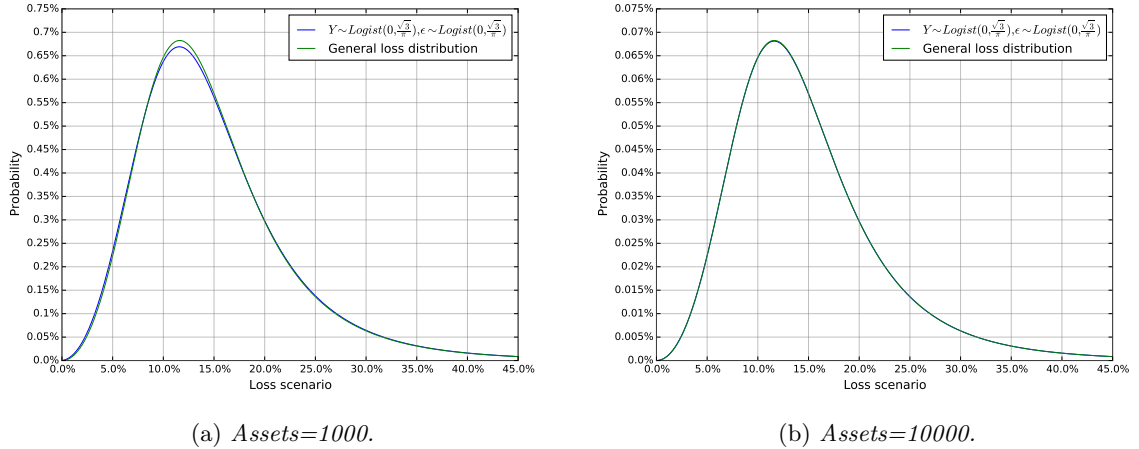

(a) *Assets=1000.*
(b) *Assets=10000.*

Figure 4: General Loss Distribution vs. Generalized Vasicek Model: $Y \sim Logistic(0, \sqrt{3}/\pi)$ and $H \sim Logistic(0, \sqrt{3}/\pi)$.

N	Mean	Stdev	$VaR_{0.999}$	CPU
1000	0.150000 (**0.0%**)	0.078573 (**1.0%**)	0.6121 (**0.3%**)	×1338
5000	0.150000 (**0.0%**)	0.077952 (**0.2%**)	0.6105 (**0.1%**)	×1472
10000	0.150000 (**0.0%**)	0.077874 (**0.1%**)	0.6103 (**0.0%**)	×1804

Table 4: Main metrics. General Loss Distribution vs. generalized Vasicek Model: $Y \sim Logistic(0, \sqrt{3}/\pi)$ and $H \sim Logistic(0, \sqrt{3}/\pi)$. Relative error (within parenthesis) with respect to the generalized Vasicek one-factor model. CPU speed up ratio compared to generalized Vasicek model.

6 Conclusions

We described the most widely used models for the calculation of default probabilities in portfolio credit risk. We introduced the Vasicek one-factor model and its generalization for factors following non Normal distributions. Similarly, we presented the large portfolio approximation method and we generated closed-form expressions for the so-called general loss distribution. In section 7 we provide code for the main routines used throughout this technical report. The code is not designed to be fast, but to serve as a guidance and point of departure for more elaborate implementations. Furthermore, the code can be easily extended to heterogeneous portfolios. As shown, only a few lines of code using the NAG Toolbox for $MATLAB^{\circledR}$ are required to implement the studied models, which makes it extraordinarily suitable for prototyping.

7 Code

7.1 NAG Functions

- `nag_stat_pdf_normal` (g01ka): probability density function of the Normal distribution.

- `nag_stat_prob_normal` (g01ea): cumulative Normal distribution function.

- `nag_stat_inv_cdf_normal` (g01fa): inverse of cumulative Normal distribution.

- `nag_stat_prob_bivariate_normal` (g01ha): lower tail probability for the bivariate Normal distribution.

- `nag_stat_prob_binomial` (g01bj): lower tail, upper tail and point probabilities associated with a binomial distribution.

- `nag_quad_1d_gauss_wgen` (d01bc): weights and abscissae for a Gaussian integration rule with a specific number of abscissae.

- `nag_specfun_tanh` (s10aa): hyperbolic tangent function.

- `nag_specfun_sinh` (s10ab): hyperbolic sine function.

- `nag_specfun_cosh` (s10ac): hyperbolic cosine function.

7.2 Main routines

7.2.1 Vasicek one-factor model

Listing 1: loss_distribution.m

```matlab
function [loss, prob_cum, prob] = loss_distribution(pd, corr, N, ...
        nodes)

loss = zeros(N, 1);
prob_cum = zeros(N, 1);
prob = zeros(N, 1);

% compute Gauss-Hermite weights and abscissae
itype = nag_int(4);
a = 0;
b = 1;
c = 0;
d = 0;
[w, x] = nag_quad_1d_gauss_wgen(itype, a, b, c, d, nag_int(nodes));

sqrt_pi = 1 / sqrt(pi);

% number of defaults = 0
prob_cum(1) = sqrt_pi * gauss_hermite(pd, corr, 0, N, w, x, nodes);
prob(1) = prob_cum(1);

for i=2:N
    loss(i) = (i-1) / N;
    prob_cum(i) = sqrt_pi * gauss_hermite(pd, corr, i-1, N, w, x,
        nodes);
    prob(i) = prob_cum(i) - prob_cum(i-1);
end
```

```matlab
1  function prob_k = gauss_hermite(pd, corr, k, N, w, x, nodes)
2
3  prob_k = 0;
4  for i=1:nodes
5      prob_k = prob_k + w(i) * binomial_default_prob(pd, corr, k, ...
6          N, x(i));
7  end
```

```matlab
1  function binomial = binomial_default_prob(pd, corr, k, N, y)
2
3  default_prob = obligor_default_prob(pd, corr, y);
4  binomial = nag_stat_prob_binomial(nag_int(N), default_prob, ...
5      nag_int(k));
```

```matlab
1  function phi = obligor_default_prob(pd, corr, y)
2
3  alpha = nag_stat_inv_cdf_normal(pd);
4  t = (alpha - sqrt(2 * corr) * y) / sqrt(1 - corr);
5  phi = nag_stat_prob_normal(t);
```

```matlab
1  function [p_mean, p_stdev] = portfolio_stats(loss, prob)
2
3  p_mean = loss' * prob;
4  p_stdev = sqrt((loss .^2)' * prob - p_mean ^ 2);
```

7.2.2 Large homogeneous portfolio

```matlab
1  function [loss, prob] = lhp_loss_distribution(pd, corr, N)
2
3  alpha = nag_stat_inv_cdf_normal(pd);
4  t = sqrt((1 - corr) / corr);
5  loss = zeros(N, 1);
6  prob = zeros(N, 1);
7
8  for i=1:N
9      loss(i) = (i-1) / N;
10     if (i-1 > 0)
11         c = nag_stat_inv_cdf_normal(loss(i));
12     else
13         c = nag_stat_inv_cdf_normal(0.01 / N);
14     end
15
16     t1 = 0.5 * c ^ 2;
17     t2 = 1 / (2 * corr);
```

```matlab
18      t3 = ( alpha − sqrt ( 1 − corr ) ∗ c ) ^ 2 ;
19      prob ( i ) = t ∗ exp ( t1 − t2 ∗ t3 ) / N;
20  end
```

```matlab
1  function [ p_mean , p_stdev ] = lhp_portfolio_stats ( pd , corr , loss , ...
2          prob )
3
4  p_mean = loss ' ∗ prob ;
5
6  norm_inv_pd = nag_stat_inv_cdf_normal ( pd ) ;
7  bivar = nag_stat_prob_bivariate_normal ( norm_inv_pd , norm_inv_pd , ...
8          corr ) ;
9  p_variance = bivar − pd ^ 2 ;
10  p_stdev = sqrt ( p_variance ) ;
```

```matlab
1  function port_quantile = lhp_quantile ( pd , corr , beta )
2
3  inv_pd = nag_stat_inv_cdf_normal ( pd ) ;
4  inv_beta = nag_stat_inv_cdf_normal ( beta ) ;
5
6  t = ( inv_pd + sqrt ( corr ) ∗ inv_beta ) / sqrt ( 1 − corr ) ;
7  port_quantile = nag_stat_prob_normal ( t ) ;
```

7.2.3 Generalized Vasicek one-factor model

```matlab
1  function [ loss , prob_cum , prob ] = logistic_loss_distribution ( ...
2          alpha , corr , N, m, nodes )
3
4  loss = zeros (N, 1 ) ;
5  prob_cum = zeros (N, 1 ) ;
6  prob = zeros (N, 1 ) ;
7
8  % compute Double exponential weights and abscissae
9  [w, x, h] = double_exponential_nodes (m, nodes ) ;
10
11  % number of defaults = 0
12  prob_cum ( 1 ) = double_exponential ( alpha , corr , 0 , N, w, x, h ) ;
13  prob ( 1 ) = prob_cum ( 1 ) ;
14
15  for i =2:N
16      loss ( i ) = ( i −1) / N;
17      prob_cum ( i ) = double_exponential ( alpha , corr , i −1, N, w, x, h ) ;
18      prob ( i ) = prob_cum ( i ) − prob_cum ( i −1) ;
19  end
```

Listing 10: double_exponential.m

```matlab
function prob_k = double_exponential(alpha, corr, k, N, w, x, h)

nodes = length(w);

prob_k = 0;
for i=1:nodes
    prob_k = prob_k + w(i) * logistic_binomial_default_prob(...
        alpha, corr, k, N, x(i));
end

prob_k = prob_k * 0.5 * pi * h;
```

Listing 11: double_exponential_nodes.m

```matlab
function [w, x, h] = double_exponential_nodes(m, n)

h_factor = -m / (m + 1);
h = n ^ h_factor;

x = zeros(2 * n, 1);
w = zeros(2 * n, 1);

for i=-n:n
    aux = 0.5 * pi * nag_specfun_sinh(i * h);
    x(i+n+1) = nag_specfun_sinh(aux);
    w(i+n+1) = nag_specfun_cosh(i * h) * nag_specfun_cosh(aux);
end
```

Listing 12: logistic_binomial_default_prob.m

```matlab
function binomial = logistic_binomial_default_prob(alpha, corr, ....
    k, N, y)

default_prob = logistic_obligor_default_prob(alpha, corr, y);
binom = nag_stat_prob_binomial(nag_int(N), default_prob, nag_int(k))
    ;
logis_pdf = logistic_pdf(y, 0, sqrt(3) / pi);
binomial = binom * logis_pdf;
```

Listing 13: logistic_obligor_default_prob.m

```matlab
function phi = logistic_obligor_default_prob(alpha, corr, y)

t = (alpha - sqrt(corr) * y) / sqrt(1 - corr);
phi = logistic_cdf(t, 0, sqrt(3) / pi);
```

7.2.4 General loss distribution

Listing 14: lhp_logistic_loss_distribution.m

```matlab
function [loss, prob] = lhp_logistic_loss_distribution(alpha, ...
        corr, N)

t = sqrt((1 - corr) / corr);

loss = zeros(N, 1);
prob = zeros(N, 1);

for i=1:N
    loss(i) = (i-1) / N;

    if (i-1 > 0)
        c = logistic_inv_cdf(loss(i), 0, sqrt(3) / pi);
        d = logistic_inv_cdf_deriv(loss(i), sqrt(3) / pi);
    else
        c = nag_stat_inv_cdf_normal(0.01 / N);
        d = logistic_inv_cdf_deriv(0.01 / N, sqrt(3) / pi);
    end

    aux = alpha / sqrt(corr) - t * c;
    t1 = logistic_pdf(aux, 0, sqrt(3) / pi);
    prob(i) = t * t1 * d / N;
end
```

Listing 15: lhp_logistic_quantile.m

```matlab
function port_quantile = lhp_logistic_quantile(alpha, corr, beta)

t = alpha - sqrt(corr) * log((1 - beta) / beta) * sqrt(3) / pi;
port_quantile = logistic_cdf(t / sqrt(1 - corr), 0, sqrt(3) / pi);
```

Listing 16: logistic_pdf.m

```matlab
function [result, ifail] = logistic_pdf(x, mu, sigma)

t = 0.5 * (x - mu) / sigma;
[d, ifail] = nag_specfun_cosh(t);
if (ifail ~= 0)
    error('Error occurred.')
end
result = 1 / (4 * sigma * d ^ 2);
```

Listing 17: logistic_cdf.m

```matlab
function [result, ifail] = logistic_cdf(x, mu, sigma)

t = 0.5 * (x - mu) / sigma;
[d, ifail] = nag_specfun_tanh(t);
if (ifail ~= 0)
    error('Error occurred.')
```

```
7  end
8  result = 0.5 * (1 + d);
```

Listing 18: logistic_inv_cdf.m

```
1  function result = logistic_inv_cdf(p, mu, sigma)
2
3  t = p / (1 - p);
4  result = log(t) * sigma + mu;
```

Listing 19: logistic_inv_cdf_deriv.m

```
1  function result = logistic_inv_cdf_deriv(x, sigma)
2
3  result = sigma / ((1 - x) * x);
```

7.3 Examples

Listing 20: example_vasicek.m

```matlab
% example Vasicek one-factor model.

pd    = 0.05;    % common probability of default
corr  = 0.10;    % common correlation between assets
N     = 100;     % number of assets
nodes = 20;      % number of quadrature nodes

% compute portfolio loss distribution
tic;
[loss, prob_cum, prob] = loss_distribution(pd, corr, N, nodes);
elapsed = toc;

% compute main portfolio metrics
[p_mean, p_stdev] = portfolio_stats(loss, prob);

% compute standard deviaton over mean
std_mean = p_stdev / p_mean;

% compute Value-at-Risk (VaR at level 99.9%)
VaR_999 = quantile(loss, prob_cum, 0.999);

% results:
fprintf('\nPortfolio metrics        \n')
fprintf('============================     \n')
fprintf('prob. default   = %10.3f\n', pd)
fprintf('correlation     = %10.3f\n', corr)
fprintf('assets          = %10.3i\n', N)
fprintf('Portfolio mean  = %10.6f\n', p_mean)
fprintf('Portfolio stdev = %10.6f\n', p_stdev)
fprintf('Stdev over mean = %10.6f\n', std_mean)
fprintf('VaR (99.9%%)     = %10.4f\n', VaR_999)
fprintf('CPU time (sec)  = %10.3f\n', elapsed)
```

```
Portfolio metrics
============================
prob. default   =      0.050
correlation     =      0.100
assets          =        100
Portfolio mean  =   0.050000
Portfolio stdev =   0.040935
Stdev over mean =   0.818697
VaR (99.9%)     =     0.2716
CPU time (sec)  =      0.023
```

```matlab
1  % example large homogeneous portfolio.
2
3  pd    = 0.15;   % common probability of default
4  corr  = 0.30;   % common correlation between assets
5  N     = 10000;  % number of assets
6
7  % compute portfolio loss distribution
8  tic;
9  [loss, prob] = lhp_loss_distribution(pd, corr, N);
10 elapsed = toc;
11
12 % compute main portfolio metrics
13 [p_mean, p_stdev] = lhp_portfolio_stats(pd, corr, loss, prob);
14
15 % compute standard deviaton over mean
16 std_mean = p_stdev / p_mean;
17
18 % compute Value-at-Risk (VaR at level 99.9%)
19 VaR_999 = lhp_quantile(pd, corr, 0.999);
20
21 % results:
22 fprintf('\nPortfolio metrics      \n')
23 fprintf('============================  \n')
24 fprintf('prob. default   = %10.3f\n', pd)
25 fprintf('correlation     = %10.3f\n', corr)
26 fprintf('assets          = %10.3i\n', N)
27 fprintf('portfolio mean  = %10.6f\n', p_mean)
28 fprintf('portfolio stdev = %10.6f\n', p_stdev)
29 fprintf('stdev over mean = %10.6f\n', std_mean)
30 fprintf('VaR (99.9%%)     = %10.4f\n', VaR_999)
31 fprintf('CPU time (sec)  = %10.3f\n', elapsed)
```

```
Portfolio metrics
==============================
prob. default   =       0.150
correlation     =       0.300
assets          =       10000
portfolio mean  =    0.150000
portfolio stdev =    0.137911
stdev over mean =    0.919408
VaR (99.9%)     =      0.7836
CPU time (sec)  =       0.020
```

```matlab
1  % example generalized Vasicek one−factor model: Logistic factor
       model.
2
3  alpha = −0.970082643277;    % threshold alpha, previously computed
4  corr  = 0.10;               % common correlation between assets
5  N     = 10000;              % number of assets
6  m     = 12;                 % double exponential parameter
7  nodes = 300;                % number of quadrature nodes
8
9  % compute portfolio loss distribution
10 tic;
11 [loss, prob_cum, prob] = logistic_loss_distribution(alpha, corr, ...
12         N, m, nodes);
13 elapsed = toc;
14
15 % compute main portfolio metrics
16 [p_mean, p_stdev] = portfolio_stats(loss, prob);
17
18 % compute standard deviaton over mean
19 std_mean = p_stdev / p_mean;
20
21 % compute Value−at−Risk (VaR at level 99.9%)
22 VaR_999 = quantile(loss, prob_cum, 0.999);
23
24 % results:
25 fprintf('\nPortfolio metrics       \n')
26 fprintf('===============================       \n')
27 fprintf('threshold       = %10.6f\n', alpha)
28 fprintf('correlation     = %10.3f\n', corr)
29 fprintf('assets          = %10.3i\n', N)
30 fprintf('Portfolio mean  = %10.6f\n', p_mean)
31 fprintf('Portfolio stdev = %10.6f\n', p_stdev)
32 fprintf('Stdev over mean = %10.6f\n', std_mean)
33 fprintf('VaR (99.9%%)     = %10.4f\n', VaR_999)
34 fprintf('CPU time (sec)  = %10.3f\n', elapsed)
```

```
Portfolio metrics
============================
prob. default   =      0.150
threshold       =  -0.970083
correlation     =      0.100
assets          =      10000
Portfolio mean  =   0.150000
Portfolio stdev =   0.077874
Stdev over mean =   0.519159
VaR (99.9%)     =     0.6103
CPU time (sec)  =     78.455
```

```matlab
1  % example large homogeneous portfolio - Logistic factor model.
2
3  pd    = 0.15;                    % common probability of default
4  alpha = -0.970082643277;         % threshold alpha, previously computed
5  corr  = 0.10;                    % common correlation between assets
6  N     = 10000;                   % number of assets
7
8  % compute portfolio loss distribution
9  tic;
10 [loss, prob] = lhp_logistic_loss_distribution(alpha, corr, N);
11 elapsed = toc;
12
13 % compute main portfolio metrics
14 [p_mean, p_stdev] = portfolio_stats(loss, prob);
15
16 % compute standard deviaton over mean
17 std_mean = p_stdev / p_mean;
18
19 % compute Value-at-Risk (VaR at level 99.9%)
20 VaR_999 = lhp_logistic_quantile(alpha, corr, 0.999);
21
22 % results:
23 fprintf('\nPortfolio metrics      \n')
24 fprintf('============================     \n')
25 fprintf('prob. default   = %10.3f\n', pd)
26 fprintf('threshold       = %10.6f\n', alpha)
27 fprintf('correlation     = %10.3f\n', corr)
28 fprintf('assets          = %10.3i\n', N)
29 fprintf('Portfolio mean  = %10.6f\n', p_mean)
30 fprintf('Portfolio stdev = %10.6f\n', p_stdev)
31 fprintf('Stdev over mean = %10.6f\n', std_mean)
32 fprintf('VaR (99.9%%)     = %10.4f\n', VaR_999)
33 fprintf('CPU time (sec)  = %10.3f\n', elapsed)
```

```
Portfolio metrics
============================
prob. default   =      0.150
threshold       =  -0.970083
correlation     =      0.100
assets          =      10000
Portfolio mean  =   0.150000
Portfolio stdev =   0.077796
Stdev over mean =   0.518639
VaR (99.9%)     =     0.6101
CPU time (sec)  =      0.040
```

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